

A Ponzano-Regge model of Lorentzian 3-Dimensional gravity

Laurent FREIDEL^{* a}

^a*Laboratoire de Physique*

Ecole Normale Supérieure de Lyon

46, allée d'Italie, 69364 Lyon Cedex 07, France[†]

Abstract

We present the construction of the partition function of 3-dimensional gravity in the Lorentzian regime as a state sum model over a triangulation. This generalize the work of Ponzano and Regge to the case of Lorentzian signature.

1 3D Gravity

It is well known that, in dimension three, gravity is a topological theory, no graviton are present and the number of effective degrees of freedom is finite dimensional. This manifests in the fact that the dynamics of Lorentzian (resp. Euclidean) gravity can be described using the topological $SO(2,1)$ (resp. $SO(3)$) BF theory (1.1) whose action is given by

$$S[B, A] = \int_M \text{Tr}(B \wedge F(A)), \quad (1.1)$$

where B is a Lie algebra valued one form, A is a gauge connection and $F(A)$ its Lie algebra valued curvature.

For this type of theory it is easy to see that the unconstrained phase space is given by a pair of ‘electric and magnetic fields’ (\bar{B}, \bar{A}) which are the restriction of the (B, A) fields on a two dimensional spatial surface. We can therefore take the wave functional as being

^{*}freidel@ens-lyon.fr

[†]URA 5672 du CNRS

functional of the connection $\psi(\bar{A})$. The main purpose of this paper is to give a computation of transition amplitude between choosen initial and final states in the lorentzian case. More precisely, lets consider a 3 dimensional (space-time) manifold M with boundary ∂M . When ∂M can be decomposed into an in and out part $\partial M = \partial M_0 \cup \partial M_1$, denoting \bar{A}_0 (resp. \bar{A}_1) the restriction of the connection A on ∂M_0 (resp. ∂M_1) it is possible to define the transition amplitude

$$\langle \Phi | \Psi \rangle_M = \int \mathcal{D}A \mathcal{D}e \bar{\Phi}(\bar{A}_0) \Psi(\bar{A}_1) e^{\frac{i}{\hbar} S[e, A]} \quad (1.2)$$

In the case where $M = \Sigma \times I$, this amplitude was computed by Witten [1]:

$$\langle \Phi | \Psi \rangle_\Sigma = \int_{\mathcal{M}_\Sigma} d\mu(A) \bar{\Phi}(A) \Psi(A). \quad (1.3)$$

Here the integral is over the moduli space of flat connections \mathcal{M}_Σ and $d\mu(A)$ is the symplectic measure on this space.

It turns out that in the case of Euclidean gravity the computation of transition amplitudes (1.2) was already obtained in the sixties by Ponzano and Regge using recoupling coefficients of $SO(3)$ [2]. Let Δ be a triangulation of the manifold M inducing on the boundary a triangulation $\bar{\Delta}_0$ (resp $\bar{\Delta}_1$) of ∂M_0 (resp. ∂M_1). Lets label the edges e of the triangulation Δ by irreducible representations of $SO(3)$ with spin $j_e \leq k$, where k is a regularisation parameter. In the triangulation Δ we distinguish two type of edges: the interior edges e which do not lie on the boundary and the boundary edges \bar{e} which are edges of the boundary triangulations $\bar{\Delta}_0$ or $\bar{\Delta}_1$. Given a tetrahedra T whose edges are labeled by representations $j_e, j_{\bar{e}}$ we denote $T(j_e, j_{\bar{e}})$ the corresponding normalized 6-j symbol. The Ponzano-Regge amplitude associated with the triangulation Δ and the coloring of the boundary triangulation $j_{\bar{e}}$ is given by:

$$Z(\Delta, j_{\bar{e}}) = \sum_{j_e} \prod_v \frac{1}{\Lambda(k)} \prod_e d(j_e) \prod_T T(j_e, j_{\bar{e}}) \quad (1.4)$$

where v denotes the vertices of the triangulation, e the interior edges, T the tetrahedra, $d(j_e)$ denotes the dimension, $\Lambda(k) \sim k^3$ is a regularisation constant and the sum is over all possible labeling of the triangulation compatible with the labeling of the boundary. The Ponzano-Regge model is obtained as the limit $k \rightarrow \infty$. Due to the identities satisfy by the 6-j symbol it turns out that the amplitude $Z(\Delta, j_{\bar{e}})$ does not depend on the choice of the triangulation Δ , but only on the boundary data $\bar{\Delta}_0, \bar{\Delta}_1$ and the coloring $j_{\bar{e}}$.

The link between Witten and Ponzano-Regge quantization can be understood using the notion of spin network. Given a triangulation $\bar{\Delta}$ of a 2-dimensional surface Σ we can construct a trivalent graph Γ dual to it. Vertices of this graph correspond to the center of the triangles and edges of Γ intersect the edges of Δ . Associated with this graph we can define a kinematical Hilbert space $V_\Gamma(\Sigma)$ which is the space of spin networks with support Γ [3]. This

means that V_Γ is the space of L^2 functions on G^E , where E denote the number of edges of Γ , invariant under the action of the gauge group acting at vertices of Γ . An orthonormal basis of this space is given by spin network functionals $\phi_{\Gamma,j} \in V_\Gamma$, where j is a coloring of the edges of Γ by irreducible representation of $SO(3)$. Given a connection A on Σ , we can construct holonomies $g_e(A)$ of the connection A along edges of the graph Γ . Taking the value of $\phi_{\Gamma,j}$ on these groups elements promotes the spin network functional into a wave function $\phi_{\Gamma,j}(\bar{A})$.

The Ponzano-Regge model gives an evaluation of the wave product (1.2) of such wave functions in term of the amplitude [4, 5]:

$$\langle \phi_{\Gamma_0,j_0} | \phi_{\Gamma_1,j_1} \rangle_\Delta := Z(\Delta, j_{\bar{e}}). \quad (1.5)$$

where Δ is a triangulation of M inducing a triangulation $\bar{\Delta}_{0,1}$ of $\partial M_{0,1}$ dual to $\Gamma_{0,1}$.

The product (1.5) coincide, up to a global normalisation, with the Witten product (1.3).

$$\langle \phi_{\Gamma_0,j_0} | \phi_{\Gamma_1,j_1} \rangle_\Delta = \langle \phi_{\Gamma_0,j_0} | \phi_{\Gamma_1,j_1} \rangle_\Sigma, \quad (1.6)$$

showing that the Ponzano Regge model is in fact equivalent to the Witten quantization. This means, for instance, that when the graphs $\Gamma_{0,1}$ do not wrap around a non contractible circle of the two dimensional surface then the amplitude is given by the “evaluation” of graphs

$$\langle \phi_{\Gamma_0,j_0} | \phi_{\Gamma_1,j_1} \rangle_\Delta = ev(\phi_{\Gamma_0,j_0}) ev(\phi_{\Gamma_1,j_1}) \quad (1.7)$$

where $ev(\Gamma, j)$ is the evaluation of the spin network, which is the value of the spin network functional $\phi_{\Gamma,j}$ on the identity group element.

1.1 Euclidean calculation

There are several arguments leading to the conclusion that the Ponzano-regge model computes transition amplitudes for 3D Euclidean gravity [4, 5]. We sketch here the argument of Ooguri [4], when there is no boundary. The main idea of this construction is to discretise the measure and the action of the partition function [6]. First, one choose a triangulation Δ of M , then to each edge e of Δ one associate a Lie algebra element B_e which corresponds to the integral of the B field along the edge and a group element g_e which corresponds to the holonomy of the connection around the edge. g_e is the product of group elements $g_{f_1} \cdots g_{f_n}$ where f_i denotes the triangles meeting at the edge e . One first integrate over the algebra elements B_e , each integration produces a delta function imposing the holonomy g_e of being the identity. It can be shown that, due to the Bianchi identity, a regularisation factor denoted Λ is needed for each vertex of the triangulation. In the case of $SU(2)$, the delta

function on the group can be expressed as a sum over the character of finite dimensional representations, by the Plancherel formula:

$$\delta(g) = \sum_j d_j \chi_j(g). \quad (1.8)$$

The sum is over all spins, d_j denotes the dimension of the spin j representation and $\chi_j(g)$ denotes the trace of a group element g in this representation.

The computation of the amplitude therefore reduces to

$$Z(\Delta) = \prod_v \Lambda \prod_f \int dg_f \prod_e \sum_{j_e} d_{j_e} \chi_{j_e}(g_e), \quad (1.9)$$

where the integral are over the group using the normalized Haar measure.

Since a face f of the triangulation possesses 3 edges, each g_f appears in three characters, so the integration over g_f involves integrals over products of three matrix elements. It is well known that such integrals are expressed in term of pairs of Clebsh-Gordan coefficients. Therefore, for each face of the triangulation we obtain a contribution involving a pair of Clebsh-Gordan, each Clebsh-Gordan being associated with a different tetrahedra. Since four faces are glued in a tetrahedra we get a pairing of four Clebsch-Gordan for each tetrahedra. In order to understand the pairing it is practical to take the spin network notation where Clebsh-Gordan coefficients correspond to trivalent vertices and the pairing is described by the tetrahedral graph. The evaluation of this graph is the normalized 6-j symbol. When there is a boundary and a spin network functional on the boundary there are additional integrations with respect to group elements h living on the boundary. If a face belongs to the boundary, one of the Clebsh-Gordan is involved in a 6-j symbol the other Clebsh-Gordan is used to contract the boundary group elements h into a spin network functional.

2 Lorentzian regime

The computation we carried out for the Euclidean case can be adapted to the Lorentzian case. The main ingredient is to reproduce the previous computation when the gauge group is no longer compact but is given by the $2+1$ Lorentz group $G = SO(2, 1) \sim SL(2, \mathbb{R})/\mathbb{Z}^2$. We recall first some fact about $SL(2, \mathbb{R})$ and its representation theory.

We saw that a key ingredient to get (1.9) was the Plancherel formula. The irreducible representation appearing in the decomposition of the delta function on $SL(2, \mathbb{R})$ are all unitary and decompose into three different series of representation:

i) The *principal series*, $T_{(i\rho-1/2, \epsilon)}$,

$\rho > 0$, $\epsilon = 0, 1/2$, $C_\rho = \rho^2 + \frac{1}{4} > \frac{1}{4}$;
 ii) The *holomorphic discrete series*, T_l^+ ,
 $l = 0, \frac{1}{2}, 1, \dots$, $C_l = -l(l+1) \leq 0$;
 iii) the *anti-holomorphic discrete series*, T_l^- ,
 $l = 0, \frac{1}{2}, 1, \dots$, $C_l = -l(l+1) \leq 0$,
 where C the Casimir.

Denoting by $\chi_{(\rho, \epsilon)}$, χ_l^\pm the corresponding characters the delta function on the group $\text{SL}(2, \mathbb{R})$ can be expressed as

$$\begin{aligned}
 (\pi)^2 \delta(g) = & \sum_l (2l+1) \{ \chi_l^+(g) + \chi_l^-(g) \} + \\
 & \sum_{\epsilon=0, 1/2} \int_0^{+\infty} d\rho \chi_{(\rho, \epsilon)}(g) \mu(\rho, \epsilon).
 \end{aligned} \tag{2.1}$$

The sum is over all positive integer and half-integer l . The Plancherel weight is given by:

$$\mu(\rho, \epsilon) = 2\rho \tanh(\pi\rho + i\epsilon\pi) \tag{2.2}$$

2.1 Evaluation of the Path integral

In order to compute the partition function we proceed as in the Euclidean case and we get a expression of the partition functional similar to the one presented in (1.9). There is however two differences. The first one is, of course, that we have to replace the sum over the spin j_e by the expansion appearing in the Plancherel formula (2.1). The second difference concerns the measure of integration. The integrand is a sum of characters and is invariant under gauge transformation, this means that there is redundant integrals in (1.9) that we have to gauge out. This was automatically taken into account in the compact case since, in that case, the volume of the group is one. The gauge symmetry appearing in (1.9) and the gauge fixing can be describe as follows. First, consider the lattice which is the one skeleton of the dual triangulation Δ^* , the edges of this lattice correspond to the faces of the triangulation and the vertices to the tetrahedra. The gauge theory we are considering is a gauge theory on this lattice. As usual, gauge variables are associated with the edges of the graph and the gauge group is acting at the vertices. In order to gauge fix the action we choose a set of link T , called a maximal tree, which does not contain any loop and which is maximal, in the sense that it cannot be extend without creating a loop. Then, we fix the value of group elements on this tree (for instance to be one) and integrate the remaining gauge variables. This integration can be obtain using the recoupling theory of $\text{SL}(2, \mathbb{R})$. The result of the computation can be presented as follows.

For each edge e of the triangulation we choose an orientation of e and a number c_e belonging to $\{-, 0, +\}$. We consider that a choice of orientation and labeling c_e is equivalent to the choice of the reverse orientation with the labeling $-c_e$. We call such a labeling of the triangulation a *causal structure* c of Δ and we say that a edge labeled by 0 is spacelike, an edge labeled by $+$ is future timelike and by $-$ is past timelike. We obtain that the partition function in the Lorentzian case is given by

$$Z(\Delta) = \sum_c Z(\Delta, c), \quad (2.3)$$

where the sum is over all causal structures c of Δ and

$$Z(\Delta, c) = \sum_{l_{e^+}} \prod_{e^+} d(l_{e^+}) \sum_{l_{e^-}} \prod_{e^-} d(l_{e^-}) \quad (2.4)$$

$$\sum_{\epsilon_0} \int d\rho_{e_0} \prod_{e_0} \mu(\rho_{e_0}, \epsilon_{e_0}) \prod_T T(l_{e^-}, l_{e^+}, \rho_{e_0}). \quad (2.5)$$

Where the sum over the holomorphic discrete series for future timelike edges, the anti-holomorphic serie for past timelike edges and over the principal series for spacelike edges. $d(l_{e^+}) = 2l_{e^+} + 1$, μ is the plancherel weight (2.2) and $T(l_{e^-}, l_{e^+}, \rho_{e_0})$ is the 6-j symbol of $SL(2, \mathbb{R})$. It cannot be defined as in the $SU(2)$ case by the contraction of four Clebsch-Gordan coefficients since such contraction would be divergent but it is defined as the matrix element of a unitary transformation on the space of invariant operators acting on the tensor product of four unitary representation of $SL(2, \mathbb{R})$. This 6-j symbol can be zero depending on admissibility conditions satisfied by the three representations labeling the edges of a triangle. When the orientation of the triangle is given from the numbering of its vertices the admissible triples are [7]:

- (l_1^+, l_2^+, l_3^+) with $l_3 > l_1 + l_2$ and $l_1 + l_2 + l_3 \in \mathbb{Z}$,
- (l_1^-, l_2^-, l_3^-) with $l_3 > l_1 + l_2$ and $l_1 + l_2 + l_3 \in \mathbb{Z}$,
- $((\rho_1, \epsilon_1), l_2^+, l_3^-)$ with $l_1 + l_2 + \epsilon \in \mathbb{Z}$,
- $((\rho_1, \epsilon_1), (\rho_2, \epsilon_2), l_3^\pm)$ with $\epsilon_1 + \epsilon_2 + l_3 \in \mathbb{Z}$,
- $((\rho_1, \epsilon_1), (\rho_2, \epsilon_2), (\rho_3, \epsilon_3))$ with $\epsilon_1 + \epsilon_2 + \epsilon_3 \in \mathbb{Z}$.

In the definition of the partition function (2.4) a regularisation and a limiting procedure is understood exactly as in the Euclidean case. We first constrain the sums and integrals to spins $l^\pm \leq k$, $\rho \leq k$, then add a regularisation factor at the vertices of the triangulation before taking the limit. The amplitudes $Z(\Delta)$ do not depend on the choice of the triangulation. This is not true in general for the causal partition function $Z(\Delta, c)$. There is however a important exception which arise if the causal structure c is such that all edges are labeled $+$ (resp. $-$). In that case the causal partition function $Z(\Delta, +)$ (resp. $Z(\Delta, -)$) is invariant under the choice of triangulation.

In Lorentzian geometry a vector in \mathbb{R}^3 can be spacelike timelike or null. This different alternatives shows up in the partition function as the possibility to choose which type of representation we assign to the edges. The fact that we assign principal series to spacelike edges and discrete series for timelike edges can be justified by several arguments which we can just outline here. First, with this geometrical understanding of representation, the admissibility rules are consistent with the rule of vector addition. Second, it is consistent with the Kirillov correspondence between representation of $SL(2, \mathbb{R})$ and orbits in R^3 since in this correspondence the discrete series are associated with two-sheeted hyperboloid which are orbit of a timelike vector.

References

- [1] E. Witten, Nucl.Phys.B 311 (1988) 46.
- [2] G. Ponzano and T. Regge, In *Spectroscopic and theoretical methods in physics*, ed. F. Block, North-Holland Amsterdam, 1968.
- [3] A. Ashtekar and J. Lewandowski, J. Diff. Geom. 17 (1995) 191.
- [4] H. Ooguri, Nucl. Phys. B382 (1992) 276.
- [5] C. Rovelli, *Phys. Rev.* **D48**, 2702. (1993).
- [6] K. Krasnov and L. Freidel, Adv. Theor. Math. Phys. 2 (1999) 1183
- [7] N. Vilenkin and A. Klimyk, Vol. 1, Mathematics and its applications, Kluwer academic publisher, 1993.